Paper presented to the 3rd IMA International Conference on Mathematics in Transport Planning and Control, Cardiff, 1-3 April 1998.

System Optimal Assignment with Departure Time Choice

J.D. Addison and B.G. Heydecker
University of London Centre for Transport Studies, University College London

ABSTRACT

This paper investigates the temporal inflow profile that minimises the total cost of travel for a single route. The problem is formulated to consider the case in which the total demand to be serviced is fixed. The approach used here is a direct calculation of the first order variation of total system cost with respect to variations in the inflow profile. Two traffic models are considered; the bottleneck with deterministic queue and the kinematic wave model. For the bottleneck model a known solution is recovered. The wave model proves more difficult and after eliminating the possibility of a smooth inflow profile the restricted case of constant inflow is solved. As the space of possible profiles is finite dimensional in this case, the standard techniques of calculus apply. We establish a pair of equations that are satisfied simultaneously by the optimal inflow and time of first departure.

1. INTRODUCTION

In this paper we investigate the temporal inflow profile to a route that gives a minimal total system cost. The cost experienced by all traffic has two components: the cost of travel along the route when account is taken of the congestion caused by this traffic and a cost associated with deviation from a desired time of arrival. We suppose that the total amount of traffic to be assigned, E, is exogenous in order to avoid the trivial solution of no travel.

Let $q(s)$ be an inflow such that the first traffic enters at time s_0. The accumulated entry flow, $A(s)$, is given by

$$A(s) = \int_{s_0}^{s} q(u)\, du \tag{1.1}$$

Except on intervals where $q(s) = 0$, $A(s)$ is a monotonic increasing function of s and is therefore inevitable. Let $s(A)$ denote this inverse: we have that $s(0) = s_0$ and $s(A)$ is the time at which the "Ath" vehicle enters the route. Similarly let $\tau(A)$ denote the time at which the accumulated efflux from the route is A. Note that τ depends on the choice of q. From (1.1) it follows that

$$\frac{dA(s)}{ds} = q(s) \text{ and so } \frac{ds(A)}{dA} = \frac{1}{q[s(A)]}. \tag{1.2}$$

Whenever the entry time function $s(A)$ is given, the inflow profile q is also available. The arrival time τ then depends on $s(A)$. When we wish to emphasise this dependence we use τ_s to denote the arrival time.

The individual cost incurred is the travel time $\tau(A)$ - $s(A)$ plus the arrival time-specific cost $f(\tau(A))$ so the total system cost, $\Theta(s)$ is (Jauffred and Bernstein, 1996 also use this form)

$$\Theta(s) = \int_0^E \tau(A) - s(A) + f\left(\tau(A)\right) dA \tag{1.3}$$

With $s(0) = s_0$ and $s(E) = T$ the relation $dA(s)/ds = q(s)$ allows a simple change of variable to give the perhaps more familiar expression in terms of flow-weighted time-specific costs:

$$\Theta(s) = \int_{s_0}^{\mathrm{T}} \left[\tau(u) - u + f\left(\tau(u)\right)\right] q(u)\, du .$$

2. Some Background

The technique used in this paper is to investigate the variation in the total cost $\Theta(s)$ for small variations $h(A)$ in the inflow profile specified by $s(A)$ for each of the bottleneck and the wave traffic models. We assume that $s(A)$ is a piecewise smooth function on $\left[0, E_0\right]$. The variations that we consider are to be sufficiently smooth that they are small in the norm $\|h\| = \max_x \left\{ |h(x)| + |h'(x)| + |h''(x)| \right\}$. For certain inflow profiles denoted by $s(A)$, we find the first order variation in $\Theta(s)$. From this, using suitable choices for the first order variation $h(A)$, we can make deductions concerning the form of the system optimal inflow profile.

3. Using the Bottleneck Model

In this section, we suppose that the route has a long free-flow section and a capacity that is determined by a single bottleneck. The free-flow travel time is denoted by ϕ and the service rate of the bottleneck by b. Given $s(A)$, we suppose that there is single period during which a queue forms. Denote by A_0 the value for which the queue begins to form and A_1 the value for which the queue disappears. Assuming that s is smooth we have $s'(A_0) = 1/b$ and $s''(A_0) < 0$. The value A_1 at which the queue disappears satisfies

$$A_1 = A_0 + b\left[s(A_1) - s(A_0)\right]. \tag{3.1}$$

The arrival time τ is given by

$$\tau_s(A) = \begin{cases} s(A) + \phi & \left(A \le A_0,\, A \ge A_1\right) \\ s(A_0) + \phi + \dfrac{(A - A_0)}{b} & \left(A_0 \le A \le A_1\right) \end{cases} \tag{3.2}$$

Now consider a small variation h of s. In what follows we will add a superior tilde $\tilde{}$ to values associated with $s + h$, so for example $\widetilde{A}_0$ and $\widetilde{A}_1$ denote the new values for which the queue appears and disappears. The new value $\widetilde{A}_0$ is given by $s'\left(\widetilde{A}_0\right)+h'\left(\widetilde{A}_0\right)=1/b$ which when expanded to first order gives

$$\widetilde{A}_0 - A_0 = -\frac{h'(A_0)}{s''(A_0)} + O\left(\left\|A_0 - A_0\right\|^2\right) + O\left(\|h\|^2\right). \tag{3.3}$$

Similarly, an expansion of (3.1) leads to

$$\widetilde{A}_1 - A_1 = \frac{h(A_1)-h(A_0)}{s'(A_0)-s'(A_1)} + O\left(\left\|A_1 - A_1\right\|^2\right) + O\left(\|h\|^2\right). \tag{3.4}$$

The two equations (3.3) and (3.4) show that the variations in $\widetilde{A}_0 - A_0$ and $\widetilde{A}_1 - A_1$ are first order in h. To find the variation in $\widetilde{\tau} = \tau_{s+h}$

$$\widetilde{\tau}(A) = \begin{cases} s(A)+h(A)+\phi & \left(A \le \widetilde{A}_0,\, A \ge \widetilde{A}_1\right) \\ s\left(\widetilde{A}_0\right)+h\left(\widetilde{A}_0\right)+\dfrac{A-\widetilde{A}_0}{b}+\phi & \left(\widetilde{A}_0 \le A \le \widetilde{A}_1\right) \end{cases} \tag{3.5}$$

There are four possibilities to consider according as $A_0 > \widetilde{A}_0$ and $A_1 > \widetilde{A}_1$. These divide the interval $[0, E]$ into five subintervals: 1. $\left[0, \min\left(A_0, \widetilde{A}_0\right)\right]$; 2. $\left[\min\left(A_0, \widetilde{A}_0\right), \max\left(A_0, \widetilde{A}_0\right)\right]$; 3. $\left[\max\left(A_0, \widetilde{A}_0\right), \min\left(A_1, \widetilde{A}_1\right)\right]$; 4. $\left[\min\left(A_1, \widetilde{A}_1\right), \max\left(A_1, \widetilde{A}_1\right)\right]$; 5. $\left[\max\left(A_1, \widetilde{A}_1\right), E\right]$. Equations (3.3) and (3.4) show that the length of intervals 2 and 4 are of order $\|h\|$. The cases that remain can separated according to the existence or non-existence of a queue. In intervals 1 and 5 there is no queue either before or after perturbation, so $\widetilde{\tau}(A) = \tau_s(A)+h(A)+O\left(\|h\|^2\right)$. In interval 3 there is a queue in both cases and the variation is found to be $\tau_{s+h}(A) = \tau_s(A)+h(A)+O\left(\|h\|^2\right)$. The variation in $\Theta(s)$ is found from

$$\Theta(s+h) = \int_0^E \tau_{s+h}(A) - s(A) - h(A) + f\left(\tau_{s+h}(A)\right) dA \tag{3.6}$$

Because the durations of the intervals between A_0 and $\widetilde{A}_0$, and A_1 and $\widetilde{A}_1$ are first order and the perturbation of the integrand is also of first order they can be ignored. Thus

$$\begin{aligned} \Theta(s+h) = &\int_0^{A_0} \tau_s(A)+h(A)-s(A)-h(A)+f\left(\tau_s(A)\right)+f'\left(\tau_s(A)\right)h(A)\, dA \;+ \\ &+\int_{A_1}^{E} \tau_s(A)+h(A)-s(A)-h(A)+f\left(\tau_s(A)\right)+f'\left(\tau_s(A)\right)h(A)\, dA + \\ &+\int_{A_0}^{A_1} \tau_s(A)+h\left(A_0\right)-s(A)-h(A)+f\left(\tau_s(A)\right)+f'\left(\tau_s(A)\right)h\left(A_0\right) dA + O\left(\|h\|^2\right) \end{aligned}$$

The first order variation is

$$\Delta\Theta(s) = \int_0^{A_0} f'\left(\tau_s(A)\right)h(A)\,dA + \int_{A_1}^{E} f'\left(\tau_s(A)\right)h(A)\,dA + \int_{A_0}^{A_1} h\left(A_0\right)-h(A)+f'\left(\tau_s(A)\right)h\left(A_0\right) dA$$

At a minimum, the variation must be positive for all choices of *h*: this is not possible unless $A_0 = 0$ and $A_1 = E$. We deduce that the outflow will always be equal to the capacity, *b*, so there will always be a queue though possibly one of zero length. The variation now becomes

$$\Delta\Theta(s) = \int_0^E \left[h(0) - h(A)\right] dA + h(0)\int_0^E f'(\tau_s(A))\, dA .$$

The queue is busy on the whole of the interval [0, E] so $s(A) \le s(0) + A / b$. If a non-zero queue arose, we would have $s(A) < s(0) + A / b$ on some interval and could then choose *h* to make $\Delta\Theta(s)$ negative while still retaining a busy queue, so we must have $s(A) = s(0) + A / b$. For this choice of *s* all variations that maintain the queue have $h(0) - h(A) \ge 0$.

At the minimum, the integral in the second term must be zero otherwise a constant variation could be chosen to make $\Delta\Theta(s)$ negative. We have $s(A) = s(0) + A / b$ so that $\tau_s(A) = s(0) + \phi + A / b$. A change of variable in this expression shows that

$$\int_0^E f'(s(0) + \phi + A / b)\, dA = b\left[f(\tau_s(E)) - f(\tau_s(0))\right].$$

Thus at the optimum $f(\tau_s(0)) = f(\tau_s(E))$ showing that the first and last traveller experience identical arrival costs. For any uniminimal arrival time cost function *f* , this determines *s*(0).

4 THE WAVE MODEL

4.1 Introduction

We next consider the Lighthill and Whitham (1955) kinematic wave model. Denote by υ the speed of traffic, which is a function of the density k, and by ω the wave speed which is also a function of the density. These speeds are related by $\omega(k) = \upsilon'(k)\,k + \upsilon(k)$. The density *k* is a function of the flow q, the two being related by $q(k) = \upsilon(k)\,k$.

The instantaneous flow into the route is given by $q(s) = 1 / \dfrac{ds}{dA} = \dfrac{dA}{ds}$. According to this model, υ and ω are functions of q which is itself a function of *A*. Thus $\upsilon'(k)$ will mean the derivative with υ' with respect to *k* and similarly with other functions.

Provided that *s*(*A*) is smooth, for given $s(A)$ we can construct a parametric representation of the accumulated efflux *G*. For the wave model we know (Newell, 1993) that for a each *A* and writing *s* for *s*(*A*), $G(s + l / \omega(k)) = A - \upsilon'(k)\,k^2\, l/\omega(k)$. Thus *G* is a parametric curve with parameter *A* which will give a single value provided that no shock-waves occur.

4.2 The Wave Model With Smooth Inflow

First we establish that a system-optimal inflow profile that is otherwise smooth must start abruptly. In this analysis we suppose that the inflow profile is smooth everywhere, in particular at time $s(0)$, and that no shock waves occur. We show that for any such inflow profile, one can be constructed that gives lower total cost.

The arrival time $\tau(A)$ satisfies $G(\tau(A)) = A$. To find $\tau(A)$, first find A_i such that, writing s_i for $s(A_i)$, we have

$$A_i - \frac{\upsilon'(q(s_i))\,k^2(q(s_i))\,l}{\omega(q(s_i))} = A. \tag{4.1}$$

Then $$\tau(A) = s_i + l/\omega(q(s_i)). \tag{4.2}$$

We now consider the effect on the total cost of a variation $h(A)$:

$$\Theta(s+h) = \int_0^E \left[\tau_{s+h}(A) - s(A) - h(A) + f(\tau_{s+h}(A))\right] dA.$$

We express τ_{s+h} in terms of τ_s and h. For fixed A let A_i be as in (4.1) and let $\tilde{A}_i$ be the corresponding value for $s+h$ with $\tilde{s}_i = (s+h)(\tilde{A}_i)$. Then $\tilde{A}_i - \dfrac{\upsilon'(\tilde{q}(\tilde{s}_i))\,k^2(\tilde{q}(\tilde{s}_i))\,l}{\omega(\tilde{q}(\tilde{s}_i))} = A$ and $\tau_{s+h}(A) = s(\tilde{A}_i) + h(\tilde{A}_i) + \dfrac{l}{\omega(\tilde{q}(\tilde{s}_i))}$. The inflow for $s+h$ is $\tilde{q}(A) = q(A) - q(A)^2\,h'(A) + O(\|h\|^2)$

so that $$\tilde{q}(A) - q(A) = -q(A)^2\,h'(A) + O(\|h\|^2) \tag{4.3}$$

To find the variation in density k we use a Taylor expansion on the right hand side of $\tilde{q} = v(\tilde{k})\tilde{k}$ to obtain $\tilde{q} = q + \omega(k)(\tilde{k} - k) + O(\|\tilde{k} - k\|^2)$. Combining this with (4.3) we get

$$\tilde{k} - k = -\frac{q^2(A)\,h'(A)}{\omega(k(A))} + O(\|\tilde{k} - k\|^2). \tag{4.4}$$

Thus $\tilde{k} - k = O(\|h\|)$. It is then straightforward to find

$$\upsilon'(\tilde{k}) = \upsilon'(k) + \upsilon''(k)(\tilde{k} - k) + O(\|h\|^2) \tag{4.5}$$

$$\omega'(\tilde{k}) = \omega'(k) + \omega''(k)(\tilde{k} - k) + O(\|h\|^2) \tag{4.6}$$

$$1/\omega'(\tilde{k}) = 1/\omega'(k) - \frac{\omega''(k)}{\omega(k)^2}(\tilde{k} - k) + O(\|h\|^2) \tag{4.7}$$

To find the variation in A_i recall from (4.1) that A_i is determined, for a suitable choice of function g, by $A = A_i + g(s'(A_i))$ and so $\tilde{A}_i$ is given by $A = \tilde{A}_i + g(s'(\tilde{A}_i) + h'(\tilde{A}_i))$. To simplify the appearance of later expressions we write s_i , $s_i{}'$ and $s_i{}''$ for the values of s, s′ and s″ evaluated at A_i. Expanding in terms of $\widetilde{A}_i - A_i$ gives

$$A = A_i + \widetilde{A}_i - A_i + g(s_i') + g'(s_i')h'(A_i) + g'(s_i')s''(A_i)\left(\widetilde{A}_i - A_i\right) + O\left(\left\|\widetilde{A}_i - A_i\right\|^2\right).$$

This then gives $$\widetilde{A}_i - A_i = -\frac{g'(s_i')h'(A_i)}{1 + g'(s_i')s_i''} + O\left(\|h\|^2\right) \tag{4.8}$$

Under our supposition that no shock waves occur, the variation $\widetilde{A}_i - A_i$ will be of order $\|h\|$.

Similarly for suitable H, $\tau(A) = s_i + H\left(s_i'\right)$ and it is a straightforward calculation to show that

$$\widetilde{\tau}(A) - \tau(A) = h(A_i) + H'(s_i')h'(A_i) + \left(s_i' + H'(s_i')s_i''\right)\left(\widetilde{A}_i - A_i\right) + O\left(\|h\|^2\right). \tag{4.9}$$

To find $g(s')$ and $H(s')$ recall that q=1/s' so $\dfrac{dq}{ds'} = -1/s'^2$. Now $g(s') = \overline{g}(q) = -\dfrac{\upsilon'(k)k^2 l}{\omega(k)}$

and $H(s') = \overline{H}(q) = \dfrac{1}{\omega(k)}$, where k is a function of q. Then $g'(s') = \overline{g}'(q)/(s')^2$ and similarly

for H. This gives $g'(s) = \dfrac{\omega'(k)l}{\left(s'\,\omega(k)\right)^3}$ and $H'(s') = -\dfrac{\omega'(k)l}{(s')^2\omega(k)^3}$ as $\upsilon(k)k = q = (s')^{-1}$. (4.10)

Substituting these into (4.8) gives $$\widetilde{A}_i - A_i = \frac{\omega'(k_i)l\,h'(A_i)}{\left(\omega(k_i)s_i'\right)^3 - \omega'(k_i)l\,s_i''}. \tag{4.11}$$

The coefficient of $\widetilde{A}_i - A_i$ in (4.9) becomes $s_i' + H'(s_i')s_i'' = \dfrac{\left(s_i'\,\omega(k_i)\right)^3 - \omega'(k_i)l\,s_i''}{s_i'^2\,\omega(k_i)^3}$ so the term

involving $\widetilde{A}_i - A_i$ simplifies to $\left(s_i' + H'(s_i')s_i''\right)\left(\widetilde{A}_i - A_i\right) = \dfrac{\omega'(k_i)l\,h'(A_i)}{s_i^2\omega(k_i)^3} = -H'(s_i')h'(A_i)$

which cancels with the preceding term giving

$$\widetilde{\tau} - \tau = h(A_i)) + O\left(\|h\|^2\right). \tag{4.12}$$

The variation in $\Theta(s)$ is then $\Delta\Theta(h) = \int_0^E h(A_i) - h(A) + h(A_i)f'(\tau(A))\,dA$. (4.13)

We can now choose a variation so that, if A_1 is the value of A_i for E

$$h(A)\ \begin{cases} = 0 & (0 \le A \le A_1) \\ > 0 & (A_1 < A \le E) \end{cases}$$

With such a choice of variation, $\Delta\Theta(h) < 0$ so that we have established by construction an inflow profile of lower cost than $\Theta(s)$. Consequently, an inflow with $q(0) = 0$ cannot be optimal, so according to the wave model, any optimal inflow will start with a step increase.

4.3 Constant inflows

In this section we investigate a restricted case of the total cost minimisation problem: we assume that the influx is at a constant rate throughout the period in during which it occurs. We require that the total throughput be equal to an exogenous value E, so there is a trade-off between inflow rate and duration, and a balance of start and end times. In this case the total system cost $\Theta(s)$ can be expressed as a function of two variables: the time s_0 at which flow starts and the rate of influx, q. The optimum for this restricted problem then satisfies

$$\frac{\partial \Theta}{\partial s_0} = 0 \quad \text{and} \quad \frac{\partial \Theta}{\partial q} = 0 \tag{4.14}$$

Because changing s_0 simply relocates the flow in time and has no effect on the travel time $\tau - s$ we have $\dfrac{d\tau}{ds_0} = 1$ and so

$$\frac{\partial \Theta}{\partial s_0} = \int_0^E f'\left(\tau(A)\right) dA . \tag{4.15}$$

For the second equation,

$$\frac{\partial \Theta}{\partial q} = \int_0^E \left[\frac{\partial \tau}{\partial q} - \frac{\partial s}{\partial q} + f'\left(\tau(A)\right) \frac{\partial \tau}{\partial q} \right] dA . \tag{4.16}$$

For a steady flow q with associated density k, there are two regions of distinct character in the time-distance diagram: an initial fan shaped region of varying density is followed by a region of constant density. The first wave of the constant density, k, leaves the route at time $\tau_1 = s_0 + l/\omega(k)$. The first traffic to experience homogeneous conditions enters at time

$$s_1(q) = s_0 + \frac{l}{\omega(k)} - \frac{l}{\upsilon(k)} . \tag{4.17}$$

Let $E_1(q)$ denote the corresponding value of the accumulated inflow so that $s(E_1(q)) = s_1(q)$. From now on we specialise to Greenshields' speed-density relationship

$$\upsilon(k) = V - \alpha k \tag{4.18}$$

In this case, the arrival time is given by

$$\tau(A) = \begin{cases} \dfrac{l}{V} + \dfrac{2\upsilon'(k)A}{V^2} + \dfrac{2\sqrt{\alpha A^2 - V l \alpha A}}{V^2} + s_0 & \left(0 \le A \le E_1(q)\right) \\ \dfrac{A}{q} + \dfrac{l}{\upsilon(k)} + s_0 & \left(E_1(q) \le A\right) \end{cases} \tag{4.19}$$

so that

$$\frac{\partial \tau}{\partial q} = \begin{cases} 0 & \left(0 \le A \le E_1(q)\right) \\ -\dfrac{A}{q^2} - \dfrac{l\,\upsilon'(k)}{\upsilon(k)^2\,\omega(k)} & \left(E_1 \le A\right) \end{cases} \qquad \text{since} \quad \frac{dk}{dq} = 1/\omega(k) \tag{4.20}$$

The departure time function is $s(A) = s_0 + \dfrac{A}{q}$ and so $\dfrac{\partial s}{\partial q} = -\dfrac{A}{q^2}$. Using this and (4.20) in (4.16), and assuming that some traffic experiences homogeneous conditions, *ie* $E_1 \le E$, gives

$$\frac{\partial\Theta}{\partial q} = \int_0^{E_1(q)} -\frac{A}{q^2}\,dA + \int_{E_1(q)}^{E} -\frac{l\,\upsilon'(k)}{\upsilon(k)^2\,\omega(k)} + \left(-\frac{A}{q^2} - \frac{l\,\upsilon'(k)}{\upsilon(k)^2\,\omega(k)}\right) f'\left(\tau(A)\right) dA$$
$$= \left.\frac{A^2}{2\,q^2}\right|_0^{E_1} - \left.\frac{A\,\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)}\right|_{E_1}^{E} - \int_{E_1}^{E}\left(\frac{A}{q^2} + \frac{l\,\upsilon'(k)}{\upsilon(k)^2\,\omega(k)}\right) f'\left(\tau(A)\right) dA \tag{4.21}$$

Note that if $E_1 \geq E$ then $\frac{\partial\Theta}{\partial q} = \int_0^{E_1(q)} \frac{A}{q^2}\,dA > 0$ so that the value of $\Theta(s)$ can be reduced by decreasing q. This is because the arrival cost is fixed but the delay will decrease as a result of reduced congestion on the route.

For A in the range $[E_1, E]$ we see from (4.19) that $\tau(A)$ is a linear function. For the change of variable $\tau = \tau(A)$ with $\tau_f = \tau(E)$ and $\tau_1 = \tau(E_1)$ the integral in (4.21) becomes

$$\int_{\tau_1}^{\tau_f} f'(\tau)\left(\frac{\tau}{q} - \frac{l}{\upsilon(k)\,q} - \frac{s_0}{q} + \frac{l\,\upsilon'(k)}{\upsilon(k)^2\,\omega(k)}\right) q\,d\tau = \int_{\tau_1}^{\tau_f} f'(\tau)\,\tau\,d\tau - \left(s_0 + \frac{l}{\omega(k)}\right)\int_{\tau_1}^{\tau_f} f'(\tau)\,d\tau\,.$$

Integrating by parts, noting that $\tau_1 = s_0 + l/\omega(k)$ and substituting into (4.21), the equation $\partial\Theta/\partial q = 0$ becomes

$$\frac{E_1^2}{2\,q^2} + \frac{E_1\,\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)} - \frac{E\,\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)} - f(\tau_f)(\tau_f - \tau_1) + \int_{\tau_1}^{\tau_f} f(\tau)\,d\tau = 0 \tag{4.22}$$

Turning our attention to $\partial\Theta/\partial s_0$ given by (4.15) we have that

$$\int_0^{E} f'\left(\tau(A)\right) dA = \int_0^{E_1} f'\left(\tau(A)\right) dA + \int_{E_1}^{E} f'\left(\tau(A)\right) dA \tag{4.23}$$

In the range of the second term $\tau(A)$ is, as we saw earlier, a linear function of A so that the same change of variable allows integration and we get

$$\int_0^{E} f'\left(\tau(A)\right) dA = \int_0^{E_1} f'\left(\tau(A)\right) dA + q\left[f(\tau_f) - f(\tau_1)\right]. \tag{4.24}$$

To proceed further, we specialise to a particular arrival-cost function. Following Vickrey (1969), we use a 2-part piecewise linear function with $m_0 < 0$, $m_1 > 0$, and ideal time of arrival 0:

$$f(t) = \begin{cases} m_0\,t & (t \leq 0) \\ m_1\,t & (t \geq 0) \end{cases} \tag{4.25}$$

Note that we must have $s_0 + \phi < 0$ otherwise (4.15) would be strictly positive so that the time of first arrival is before the ideal one. The cases with $\tau_1(q)$ negative and positive are now considered separately. In the former case, $f'(\tau(A)) = m_0$ on $[0, E_1(q))$ and we get from (4.25) and (4.14) the equation for $\partial\Theta/\partial s_0 = 0$ which, together with the special form (4.25) gives

$$m_0\,E_1(q) + q\,m_1\,\tau_f - q\,m_0\,\tau_1 = 0\,. \tag{4.26}$$

Because $E_1(q)$ is the first traffic to experience homogeneous conditions, we find that for Greenshields' speed-density function $E_1(q) = \left(V - \omega(k)\right)^2 / \left(4\,\alpha\,\omega(k)\right)$. Using this, $\tau_1(q) = s_0 + l/\omega(k)$ and $\tau_f = s_0 + E/q + l/\upsilon(k)$ we obtain

$$s_0 = \frac{1}{m_0 - m_1}\left(\frac{m_0\,E + m_1\,E_1}{q} + \frac{m_1\,l}{\upsilon(k)} - \frac{m_0\,l}{\omega(k)}\right) \tag{4.27}$$

Now consider the case $\tau_1(q) \geq 0$. Let E_2 be the index of the traffic that arrives at the destination at the desired arrival time 0. The integral in (4.25) evaluates to $m_0 E_2 + m_1(E_1 - E_2)$ and (4.25) becomes

$$(m_0 - m_1)\,E_2 + m_1\,E_1 + q\,m_1\,(\tau_f - \tau_1) = 0\,. \tag{4.28}$$

During the initial phase when the density at the exit varies the accumulated efflux is given by $G(\tau) = \left(V(\tau - s_0) - l\right)^2 / \left(4\,\alpha\,(\tau - s_0)\right)$; E_2 is the value when $\tau = 0$ so that

$$E_2 = -\frac{\left(V\,s_0 + l\right)^2}{4\,\alpha\,s_0}\,. \tag{4.29}$$

Also E_1 is the accumulated flow at the end of this period of variable efflux. The end is at time $\tau_1 = s_0 + l/\omega(k)$ so that

$$E_1 = \frac{l\left(V - \omega(k)\right)^2}{4\,\alpha\,\omega(k)} = \frac{\alpha\,k^2 l}{\omega(k)}\,. \tag{4.30}$$

We note that E_2 is independent of q. Using $\tau_f = E/q + l/\upsilon(k)$ and $\tau_1 = s_0 + l/\omega(k)$, substituting the explicit expressions for $\omega(k)$ and $\upsilon(k)$ shows that the expression $m_1E_1 + qm_1(\tau_f - \tau_1) + m_1E$ is independent of k and hence of q. This substitution leads to the quadratic equation

$$\left(m_1 - m_0\right)V^2 s_0^2 + \left[2\,V\,l\left(m_1 - m_0\right) - 4\,\alpha\,m_1\,E\right]s_0 + \left(m_1 - m_0\right)\omega(k)\,l^2 = 0\,. \tag{4.31}$$

Similarly (4.23) gives rise to a pair of equations:

$$\frac{E_1^2}{2\,q^2} - \frac{\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)}\left(E - E_1\right) - \frac{m_1}{2}\left(\tau_f - \tau_1\right)^2 - \frac{m_0 - m_1}{2}\tau_1^2 = 0 \qquad \left(\tau_1(q) \leq 0\right) \tag{4.32}$$

$$\frac{E_1^2}{2\,q^2} - \frac{\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)}\left(E - E_1\right) - \frac{m_1}{2}\left(\tau_f - \tau_1\right)^2 = 0 \qquad \left(\tau_1(q) \geq 0\right). \tag{4.33}$$

Gathering together the various equations that arise from $\dfrac{\partial\Theta}{\partial s_0} = 0$ and $\dfrac{\partial\Theta}{\partial q} = 0$ give the specification of the system optimal inflow rate and start time for the present restricted problem for a single route with specified total throughput. This the optimal assignment is given in the case that $\tau_1(q) \leq 0$ by (4.27) and (4.32) as

$$\begin{gathered}\frac{E_1^2}{2\,q^2} - \frac{\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)}\left(E - E_1\right) - \frac{m_1}{2}\left(\tau_f - \tau_1\right)^2 - \frac{m_0 - m_1}{2}\tau_1^2 = 0 \\ s_0 - \frac{m_1}{m_0 - m_1}\left(\frac{E}{q} + \frac{l}{\upsilon(k)}\right) + \frac{m_0}{m_0 - m_1}\left(\frac{E_2}{q} - \frac{l}{\omega(k)}\right) = 0\end{gathered} \tag{4.34}$$

and case that $\tau_1(q) \geq 0$ by (4.31) and (4.33) as

$$\begin{gathered}\frac{E_1^2}{2\,q^2}-\frac{\upsilon'(k)\,l}{\upsilon(k)^2\,\omega(k)}\left(E-E_1\right)-\frac{m_1}{2}\left(\tau_f-\tau_1\right)^2=0\\ \left(m_1-m_0\right)V^2 s_0^2+\left[2\,V\,l\left(m_1-m_0\right)-4\,\alpha\,m_1\,E\right]s_0+\left(m_1-m_0\right)\omega(k)\,l^2=0\,.\end{gathered}\tag{4.38}$$

4. CONCLUSIONS

The analysis presented in this paper has established several results that describe the assignment of a specified amount of traffic to a single route that minimises the sum of the total travel and arrival-time costs. When the bottleneck model is used to describe a congested route, the optimal inflow rate is equal to the capacity so that the outflow rate is maximal but queueing does not occur; the first departure is timed so that the arrival-time cost incurred by the first and the last departures are identical. When the more detailed kinematic wave model is used, the optimal inflow rate is shown to have a step increase from 0 at the time of first departure. Explicit equations have been established for the optimal constant inflow rate and associated time of first departure when Vickrey's 2-part piecewise linear arrival cost function is adopted.

ACKNOWLEDGEMENTS

The authors are grateful to Richard Allsop for his encouragement and discussions on the topic of this paper. Thanks are also due to an anonymous referee for his helpful comments. This work was funded by the UK Engineering and Physical Science Research Council.

REFERENCES

JAUFFRED, F.J., and BERNSTEIN, D. (1996) An alternative Formulation of the Simultaneous Route and Departure Time Choice Equilibrium Problem. *Transportation Research*, **4C**(6), 339-357.

LIGHTHILL, M.J. and WHITHAM, G.B. (1955) On kinematic waves: II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society*, **229A**, 317-45.

NEWELL, G.F. 1993. A simplified theory of kinematic waves in highway traffic, Part I: General theory. *Transportation Research*, **27B**(4), 281-7.

VICKREY, W.S. (1969) Congestion theory and transport investment. *The American Economic Review*, **59**, 251-61.